Mathematica File: Instructions and Explanations

This computes beliefs for up to N=7 states, starting in the middle ($n_0 = 4$), and then optimal cutoffs including for truncated systems N=2, N=3, N=4, N=5, and finally N=7 (warning: N=7 is slow!!). Beliefs are in terms of threshold *likelihood ratios* that you may specify, and parameters η, ξ, assuming symmetric prior $\mu = 1$. ****IMPORTANT**: thresholds below n0 are ordered low-high, as P1<P2<P3<1; and thresholds above n0 are ordered high-low, as p1>p2>p3>1. You must have P3<1<p3 if you want to start in n0=4, otherwise there may be no/multiple solutions.** You are welcome to collapse intervals to get a smaller system, this works for up to 3 states below initial and up to 3 above. For example for $n_0 = 2$ and $N = 5$ (one state below initial and three above), you need to specify one lower cutoff P3<1 and 3 upper cutoffs p1>p2>p3>1, then collapse the bottom two intervals via P1=P2=P3.

The optimization section uses action bias $\lambda \equiv \pi/\Pi$ and assumes (via the choice of n_0) that this is at most 1. It gets exponentially slower as the # memory states N rises, so I have explicitly programmed in smaller systems: first $N = 2$ (this is instant), then $N = 3$ (about 5 seconds), $N = 4$ (30 seconds), $N = 5$ (2 minutes), and $N = 7$ (8 minutes). I skipped N=6.

1 How to Use Mathematica File

I've bunched quite a few command lines together, but you still need to enter command lines at key spots. Look for large gaps where you need to hit shift+enter to execute the commands:

1. The first part of the file computes beliefs (likelihood ratios) outside $n_0 = 4$. Hit shift+enter at the end (after the documentation note following command QHavg3)

2. The next part of the file computes the belief (likelihood ratio) in initial state $n_0 = 4$, then computes the ρ^θ vectors, and finally computes the full vector $(Q_1, Q_2, .., Q_6, Q_7)$ of belief likelihood ratios for N=7 states, given your specified cutoffs and parameters η, ξ. Look for the documentation note "this calculates the full Q vector starting in 4" and hit shift+enter after this. Then, you can try or edit the example that computes the full belief vector for cutoffs P1=.1,P2=.5,P3=.9, p1=5, p2=2, p3=1.1, $\eta = .3, \xi = 4$.

3. The next part of the file is not super interesting as a standalone, but computes key payoff terms that we'll need to optimize. Scroll past the W's and the payoff command, to the documentation note "This is the scaled payoff, assuming we start in n0=4" and hit shift+enter at the end of this paragraph. Then, you can try or edit the example that computes the (scaled) payoff for the example cutoffs in point 2 above.

4. Next part computes the optimal cutoff for N=2 memory states. Hit shift+enter after the command Popt2state. Popt2[η, ξ, λ] finds the optimal cutoff for parameters η, ξ, and $\lambda \equiv \pi/\Pi \leq 1$. This is followed by an example you can try or edit, finding the optimal cutoff for N=2 when $\eta = .3, \xi = 4, \lambda \equiv \pi/\Pi = 0.5$.

5. Next part computes the optimal cutoffs for N=3 memory states. Hit shift+enter after command Popt3state. This is again followed by an example you can try or edit.

6. And similarly, hit shift+enter after the commands Popt4 (optimal cutoffs for an N=4 system), Popt5 (for N=5), and Popt7 (for N=7).

2 Explanations of Belief Equations

The mathematica file mostly uses the paper notation, with the following poor notation caveat: in the Mathematica file we index states below n_0 from the bottom, and above n_0 from the top, with H meaning "high state" ie above n_0 and L meaning "low state" ie below n_0. So (Q1,Q2,Q3)=(QL1,QL2,QL3) (low states below initial: 1st from bottom, 2nd from bottom, 3rd from bottom); then $Q4 = Q_{n_0}$ is the initial state belief; then (Q5,Q6,Q7)=(QH3,QH2,QH1) (high states above initial: 3rd from top, 2nd from top, 1st from top).

Everything involves parameters η and ξ, and uses the following functions specified at the start of Appendix C in the paper:

$$\beta(\gamma) \equiv \left(\frac{\xi-1}{\xi+1}\right)\frac{\left(1-\gamma^{\frac{1}{2}(\xi+1)}\right)}{\left(1-\gamma^{\frac{1}{2}(\xi-1)}\right)} \tag{1}$$

$$f(R) = \left(\frac{\eta}{1-\eta}\left(R-\frac{\xi-1}{\xi+1}\right)+\frac{2}{\xi+1}(R)^{\frac{1}{2}(\xi+1)}\right) \tag{2}$$

$$g(R,\gamma) = \frac{\frac{2\eta\xi}{(1-\eta)(\xi+1)}(\beta-R)+\left(\frac{\gamma}{R}\right)^{\frac{1}{2}(\xi-1)}\left(\beta-\frac{\xi-1}{\xi+1}\gamma\right)-R^{\frac{1}{2}(\xi+1)}\left(1-\frac{\xi-1}{\xi+1}\beta\right)}{\left(\frac{\beta^2}{\gamma}-1\right)\left(\left(\frac{1}{\gamma}\right)^{\frac{1}{2}(\xi-1)}-1\right)} \tag{3}$$

Notation otherwise follows the paper: ρ_i^θ is the chance of being in state i when the final amnesia shock hits, conditional on θ. $Q_i \equiv \rho_i^H/\rho_i^L$ is the belief likelihood ratio in state i (using symmetric prior $\mu = 1$). And $\phi_i \equiv \rho_i^L/\rho_1^L$ and $\Phi_i \equiv \rho_i^H/\rho_N^H$.

- Belief (likelihood ratios) in states below n_0 are computed recursively by $\phi_1 = 1$, and Q_1 solves $f\left(\frac{Q_1}{P_1}\right) = 0$. Then for $2 \le i < n_0$,

$$\sum_{k\le i-1}\phi_k\left(\frac{Q_k}{P_i}\right)^{\frac{1}{2}(\xi+1)}+\left(\underbrace{\frac{\sum_{k\le i-1}\phi_k f\left(\frac{Q_k}{P_i}\right)}{f\left(\frac{Q_i}{P_i}\right)}}_{-\phi_i}\right)g\left(\frac{Q_i}{P_i},\frac{P_{i-1}}{P_i}\right)=0 \text{ and } \phi_i=-\frac{\sum_{k\le i-1}\phi_k f\left(\frac{Q_k}{P_i}\right)}{f\left(\frac{Q_i}{P_i}\right)} \tag{4}$$

- Belief (likelihood ratios) in states above n_0 are computed recursively by $\Phi_N = 1$, and Q_N solves $f\left(\frac{P_{N-1}}{Q_N}\right) = 0$. Then for $n_0 < i \le N-1$, Q_i solves the first equation below and Φ_i is given by the second:

$$\sum_{k\ge i+1}\frac{\rho_k^H}{\rho_N^H}\left(\frac{P_{i-1}}{Q_k}\right)^{\frac{1}{2}(\xi+1)}+\left(\underbrace{\frac{\sum_{k\ge i+1}\frac{\rho_k^H}{\rho_N^H}f\left(\frac{P_{i-1}}{Q_k}\right)}{f\left(\frac{P_{i-1}}{Q_i}\right)}}_{\equiv-\Phi_i}\right)g\left(\frac{P_{i-1}}{Q_i},\frac{P_{i-1}}{P_i}\right)=0 \text{ and } \Phi_i=-\frac{\sum_{k\ge i+1}\Phi_k f\left(\frac{P_{i-1}}{Q_k}\right)}{f\left(\frac{P_{i-1}}{Q_i}\right)} \tag{5}$$

- For initial state belief Q_{n_0}, let $\overline{Q}_{n_0-1}$ be the average belief Q_j in states $j \leq n_0 - 1$, conditional on $\theta = L$; and similarly define $\underline{Q}_{n_0+1}$ as the average belief Q_j in state $j \geq i_0 + 1$, given $\theta = L$. In terms of the ϕ_j, Φ_j, Q_j expressions above,

$$\overline{Q}_{n_0-1} = \frac{\sum_{j \leq n_0-1} \phi_j Q_j}{\sum_{j \leq n_0-1} \phi_j}, \text{ and } \underline{Q}_{n_0+1} = \frac{\sum_{j \geq n_0+1} \Phi_j}{\sum_{j \geq n_0+1} \frac{\Phi_j}{Q_j}} \tag{6}$$

Now define the following variables, all functions of these averages, where $\gamma_{n_0-1} = P_{n_0-1}/P_{n_0}$:

$$(X_1, X_2, X_3, X_4) = \left(\left(\frac{\underline{Q}_{n_0+1}}{P_{n_0}} - \frac{\xi+1}{\xi-1} \right) \gamma_{n_0-1}^{\frac{1}{2}(\xi-1)}, \frac{\xi+1}{\xi-1} \frac{\underline{Q}_{n_0+1}}{P_{n_0}} - 1, \frac{\xi+1}{\xi-1} - \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}}, \gamma_{n_0-1}^{\frac{1}{2}(\xi+1)} \left(1 - \frac{\xi+1}{\xi-1} \frac{\overline{Q}_{n_0-1}}{P_{n_0-1}} \right) \right) \tag{7}$$

- Then, the initial state belief Q_{i_0} is the root of the following equation:

$$\left[X_1 - \frac{\left(\underline{Q}_{n_0+1} - 1 \right)}{1 - \overline{Q}_{n_0-1}} X_3 \right] \left(\frac{Q_{n_0}}{P_{n_0}} \right)^{\frac{1}{2}(\xi+1)} + \left[X_2 - \frac{\left(\underline{Q}_{n_0+1} - 1 \right)}{1 - \overline{Q}_{n_0-1}} X_4 \right] \left(\frac{P_{n_0-1}}{Q_{n_0}} \right)^{\frac{1}{2}(\xi-1)}$$
$$+ \frac{\eta}{1-\eta} \frac{\xi-1}{2} \frac{1 - Q_{n_0}}{1 - \overline{Q}_{n_0-1}} \left(X_2 X_3 - X_1 X_4 \right) \tag{8}$$

3 Explanation of Optimization Part

- for the mathematica segment involving the W's, WHi is the terminal payoff if the last amnesia shock hits in memory state i, given $\theta = H$; divided by Π. (Comparing to paper notation, $WHi = w_i^H/\Pi$). And $WLi \equiv w_i^L/\pi$
- then, each of the optimization sections (for N=2, N=3, N=4, N=5, N=7) first writes out the equations that define the relevant post-amnesia payoff gaps. For example if $N = 7$, the indifference FOC's require computing 6 gaps in each true state θ, namely $\Delta_{12}^\theta, \Delta_{23}^\theta, ..., \Delta_{67}^\theta$, where $\Delta_{i,i+1}^\theta \equiv \left| \nu_{i+1}^\theta - \nu_i^\theta \right|$. So we begin by computing these gaps as a function of cutoffs, assuming Bayesian beliefs for these cutoffs. Then, the indifference FOC's that determine optimal thresholds, ie $P_i \Delta_{i,i+1}^H / \Delta_{i,i+1}^L = 1$ for the optimal cutoff (likelihood ratio) P_i, and finally Popt2, Popt3, Popt4, etc computes the optimal cutoffs if N=2,N=3,N=4.
  - Note that I specified search ranges for the thresholds; these work for many parameters but may eventually require adjustment

4 Derivation of Continuation Payoff Gaps

In each optimization segment, the mathematica file involves a set of equations labeled eg l3a, h3a, and says these are the equations that define the contination payoff gaps $\Delta_{i,i+1}^\theta$. These equations were derived as follows: The basic recursion for ν_i^θ is

$$\begin{aligned} \nu_i^\theta &= \eta w_i^\theta + (1-\eta) \sum_{j \leq i-1} \lambda_{i,j}^\theta v_j^\theta + (1-\eta) \sum_{j \geq i+1} \lambda_{i,j}^\theta v_j^\theta + (1-\eta) \left(1 - \sum_{j \leq i-1} \lambda_{i,j}^\theta - \sum_{j \geq i+1} \lambda_{i,j}^\theta \right) v_i^\theta \\ &\Leftrightarrow \eta \nu_i^\theta = \eta w_i^\theta + (1-\eta) \sum_{j \leq i-1} \lambda_{i,j}^\theta \left(\nu_j^\theta - \nu_i^\theta \right) + \sum_{j \geq i+1} \lambda_{i,j}^\theta \left(\nu_j^\theta - \nu_i^\theta \right) \end{aligned} \tag{9}$$

To make this computationally less tedious, we rewrite using cumulative transition chances and incremental post amnesia payoff gaps. As in the proof of Proposition 1, define (for $i \leq j$) $b^{\theta}_{i,j}$ as the chance of moving up from i to j or above, and (for $i > j$) $a^{\theta}_{i,j}$ as the chance of moving i to j or below. With this, (9) rearranges to

$$\frac{\eta}{1-\eta}\nu^{\theta}_{i} = \frac{\eta}{1-\eta}w^{\theta}_{i} + \sum_{j \geq i+1} b^{\theta}_{i,j}(\nu^{\theta}_{j} - \nu^{\theta}_{j-1}) - \sum_{j \leq i-1} a^{\theta}_{i,j}(\nu^{\theta}_{j+1} - \nu^{\theta}_{j})$$

Taking differences in state $i+1$ vs i, letting $\Delta^{\theta}_{i,i+1} = \left|\nu^{\theta}_{i+1} - \nu^{\theta}_{i}\right|$, this gives

$$\left[\frac{\eta}{1-\eta} + b^{\theta}_{i,i+1} + a^{\theta}_{i+1,i}\right]\Delta^{\theta}_{i,i+1} = \frac{\eta}{1-\eta}(w^{\theta}_{i+1} - w^{\theta}_{i}) + \sum_{j \geq i+2}\left(b^{\theta}_{i+1,j} - b^{\theta}_{i,j}\right)\Delta^{\theta}_{j-1,j} + \sum_{j \leq i-1}\left(a^{\theta}_{i,j} - a^{\theta}_{i+1,j}\right)\Delta^{\theta}_{j,j+1} \tag{10}$$

Now, recall that $b^{H}_{i,j} = \frac{\xi+1}{2\xi}\left(\frac{Q_i}{P_{j-1}}\right)^{\frac{1}{2}(\xi-1)}$, $a^{H}_{i,j} = \frac{\xi-1}{2\xi}\left(\frac{P_j}{Q_i}\right)^{\frac{1}{2}(\xi+1)}$, in state L just interchange $\xi-1$ and $\xi+1$. Thus, we can simplify (10) using the following:

$$\begin{aligned}
\sum_{j \geq i+2}\left(b^{H}_{i+1,j} - b^{H}_{i,j}\right)\Delta^{H}_{j-1,j} &= \frac{\xi+1}{2\xi}\left(Q_{i+1}^{\frac{1}{2}(\xi-1)} - Q_{i}^{\frac{1}{2}(\xi-1)}\right)\sum_{j \geq i+2}\frac{\Delta^{H}_{j-1,j}}{(P_{j-1})^{\frac{1}{2}(\xi-1)}}, \text{ and} \\
\sum_{j \leq i-1}\left(a^{H}_{i,j} - a^{H}_{i+1,j}\right)\Delta^{H}_{j,j+1} &= \frac{\xi-1}{2\xi}\left(Q_{i}^{-\frac{1}{2}(\xi+1)} - Q_{i+1}^{-\frac{1}{2}(\xi+1)}\right)\sum_{j \leq i-1}P_j^{\frac{1}{2}(\xi+1)}\Delta^{H}_{j,j+1}
\end{aligned}$$

Plugging into (10) yields the following recursion:

$$\begin{aligned}
&\left[\frac{\eta}{1-\eta} + \frac{\xi+1}{2\xi}\left(\frac{Q_i}{P_i}\right)^{\frac{1}{2}(\xi-1)} + \frac{\xi-1}{2\xi}\left(\frac{P_i}{Q_{i+1}}\right)^{\frac{1}{2}(\xi+1)}\right]\Delta^{H}_{i,i+1} \\
= \;& \frac{\eta}{1-\eta}(w^{H}_{i+1} - w^{H}_{i}) + \frac{\xi+1}{2\xi}\left(Q_{i+1}^{\frac{\xi-1}{2}} - Q_{i}^{\frac{\xi-1}{2}}\right)\sum_{j \geq i+2}\frac{\Delta^{H}_{j-1,j}}{(P_{j-1})^{\frac{\xi-1}{2}}} + \frac{\xi-1}{2\xi}\left(Q_{i}^{-\frac{\xi+1}{2}} - Q_{i+1}^{-\frac{\xi+1}{2}}\right)\sum_{j \leq i-1}P_j^{\frac{\xi+1}{2}}\Delta^{H}_{j,j+1}
\end{aligned}$$

And in state L, simply replace $\xi+1$ with $\xi-1$ and vice versa, and change $w^{H}_{i+1} - w^{H}_{i}$ to $w^{L}_{i} - w^{L}_{i+1}$. In all of these equations, the Q_i's are the Bayesian beliefs as a function of the cutoffs and parameters.